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LETTERS TO THE EDITOR

FUNDAMENTAL FREQUENCY OF TRANSVERSE VIBRATION OF AN ANNULAR PLATE OF CYLINDRICAL ANISOTROPY AND BILINEARLY VARYING THICKNESS

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1. INTRODUCTION

A recent technical note [1] deals with the analysis of transverse vibrations of circular annular plates of cylindrical anisotropy with a simply supported or clamped outer edge and a free inner boundary for two situations of variable thickness: discontinuous variation and linearly varying thickness. More recently the authors came upon the problem of determining the fundamental frequency in the case of bilinearly varying thickness; see Figure 1. Since the problem does possess practical interest from an engineering viewpoint a brief treatment of its approximate analytical solution and numerical values of the fundamental frequency coefficients are presented here.

2. APPROXIMATE ANALYTICAL SOLUTION

In the case of normal modes of vibration the problem is described by the energy functional [2]

$$J(W) = \iint_p \left[D_r(\bar{r}) W''^2 + D_\theta(\bar{r}) \left(\frac{W'}{\bar{r}} \right)^2 + 2D_r(\bar{r}) \upsilon_\theta \frac{W'W''}{\bar{r}} \right] \bar{r}\, \mathrm{d}\bar{r}\, \mathrm{d}\theta$$

$$- D_r(a) 2\pi a \left[W''(a) + \upsilon_\theta \frac{W'(a)}{a} \right] W'(a) - \rho\omega^2 \iint_p h(\bar{r}) W^2 \bar{r}\, \mathrm{d}\bar{r}\, \mathrm{d}\theta, \tag{1}$$

subject to the boundary conditions at the outer edge:

$$W(a) = 0, \qquad W'(a) = -\phi D_r(a) \left[W''(a) + \upsilon_\theta \frac{W'(a)}{a} \right] \tag{2a, b}$$

The natural boundary conditions at the inner edge will not be taken into account. Accordingly, co-ordinate functions which are commonly used in the case of solid circular plates elastically restrained against rotation at the outer edge will be employed [3].

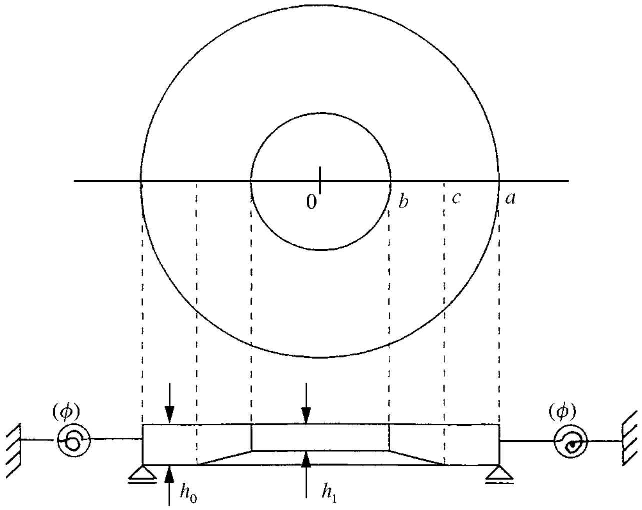

Figure 1. Vibrating structural system under study.

Introducing the dimensionless variable $r = \bar{r}/a$ and substituting in equations (1) and (2) one obtains

$$\frac{a^2}{2\pi D_{r_0}} J(W) = \int_{r_b}^{1} g^3(r)\left[W''^2 + \frac{D_{\theta_0}}{D_{r_0}}\frac{W'^2}{r^2} + 2\upsilon_\theta \frac{W'W''}{r}\right] r\,\mathrm{d}r$$

$$- [W''(1) + \upsilon_\theta W'(1)]\,W'(1) - \Omega^2 \int_{r_0}^{1} g(r) W^2 r\,\mathrm{d}r, \qquad (3)$$

$$W(1) = 0, \qquad W'(1) = -\,\phi'[W''(1) + \upsilon_\theta W'(1)], \qquad (4a, b)$$

where

$$D_r = D_{r_0} g^3(\bar{r}), \qquad D_\theta = D_{\theta_a} g^3(r), \qquad D_{r_0} = \frac{E_r h_0^3}{12(1-\upsilon_r\upsilon_\theta)}, \qquad D_{\theta_a} = \frac{E_\theta h_0^3}{12(1-\upsilon_r\upsilon_\theta)},$$

$$g(r) = \begin{cases} \dfrac{1-e}{r_b - r_c}(r - r_b) + e, & r_b \leqslant r \leqslant r_c, \\ 1 & r_c \leqslant r \leqslant 1, \end{cases}$$

$$r_b = \frac{b}{a}, \qquad r_c = \frac{c}{a}, \qquad h(r) = h_0 g(r), \qquad e = \frac{h_1}{h_0}, \qquad \Omega^2 = \frac{\rho h_0 a^4}{D_{r_0}}\omega^2, \qquad \phi' = \frac{\phi D_{r_0}}{a}.$$

The displacement amplitude is expressed now in terms of the summation of polynomial co-ordinate functions

$$W_a = \sum_{j=1}^{N} C_j \varphi_j(r) = \sum_{j=1}^{N} C_j(a_j r^{p+j-1} + b_j r^{j+1} + 1), \qquad (5)$$

where "p" is Rayleigh's optimization parameter [4].

TABLE 1

Fundamental frequency coefficients of circular annular plates of cylindrical anisotropy of bilinearly varying thickness and simply supported at the outer edge

		$e = 0{\cdot}8$				$e = 0{\cdot}6$			
$D_{\theta 0}/D_{r0}$	r_b	$r_c = 0{\cdot}2$	0·4	0·6	0·8	0·2	0·4	0·6	0·8
0·50	0	3·770	3·423	3·243	3·052	3·510	3·027	2·784	2·349
	0·2		3·061	2·852	2·663		2·976	2·688	2·264
	0·4			2·983	2·679			3·040	2·493
	0·6				3·514				3·645
0·75	0	4·399	3·973	3·720	3·424	4·186	3·705	3·384	2·756
	0·2		3·817	3·540	3·273		3·744	3·484	2·816
	0·4			3·775	3·383			3·854	3·157
	0·6				4·456				4·624
1	0	4·800	4·439	4·131	3·746	4·715	4·268	3·888	3·090
	0·2		4·415	4·082	3·744		4·362	3·950	3·255
	0·4			4·421	3·954			4·522	3·699
	0·6				5·230				5·429
1·25	0	5·195	4·849	4·498	4·033	5·155	4·758	4·330	3·375
	0·2		4·918	4·541	4·138		4·890	4·438	3·627
	0·4			4·979	4·445			5·102	4·167
	0·6				5·903				6·129
1·50	0	5·543	5·220	4·834	4·297	5·537	5·196	4·728	3·625
	0·2		5·357	4·944	4·480		5·355	4·873	3·953
	0·4			5·476	4·881			5·621	4·585
	0·6				6·505				6·756

Application of the classical Rayleigh–Ritz method leads to the following linear system of equations in the C_j's:

$$\frac{a^2}{4\pi D_{r_0}}\frac{\partial J}{\partial C_i} = \left\{\sum_{j=1}^{N}\int_{r_b}^{1} g^3(r)\left[\varphi_j''\varphi_i'' + \frac{D_{\theta_0}}{D_{r_0}}\frac{\varphi_j'\varphi_i'}{r^2} + \upsilon_\theta\frac{\varphi_j''\varphi_i' + \varphi_j'\varphi_i''}{r}\right] r\,\mathrm{d}r \right.$$
$$- \frac{1}{2}\sum_{1}^{N}\left[\varphi_j'(1)(\varphi_i''(1) + \upsilon_\theta\varphi_i'(1)) + (\varphi_j''(1) + \upsilon_\theta\varphi_j'(1))\varphi_i'(1)\right]$$
$$\left. - \Omega^2\int_{r_b}^{1} g(r)\varphi_j\varphi_i r\,\mathrm{d}r\right\} C_j = 0, \quad (i, j = 1, 2, \ldots, N). \tag{6}$$

The non-triviality condition yields a secular determinant whose lowest root is the fundamental frequency coefficient Ω_1 which is minimized with respect to "p" in order to optimize the value of Ω_1.

3 NUMERICAL RESULTS

All the numerical determinations were made for $\upsilon_\theta = 0{\cdot}30$ and $N = 7$. A good rate of convergence was observed from a practical engineering viewpoint and no significant change was observed as N was increased from $N = 6$ to 7.

TABLE 2

Fundamental frequency coefficients of circular annular plates of cylindrical anisotropy of bilinearly varying thickness and clamped at the outer edge

		$e = 0{\cdot}8$				$e = 0{\cdot}6$			
$D_{\theta 0}/D_{r0}$	r_b	$r_c = 0{\cdot}2$	0·4	0·6	0·8	0·2	0·4	0·6	0·8
0·50	0	8·995	9·190	9·356	8·056	8·849	9·914	9·234	5·099
	0·2		9·441	9·478	8·070		10·238	9·442	5·279
	0·4			13·953	12·033			16·039	8·648
	0·6				27·609				28·871
0·75	0	9·574	9·624	9·729	8·432	9·510	10·328	9·573	5·319
	0·2		10·000	10·029	8·699		10·721	10·057	5·870
	0·4			14·289	12·421			16·324	9·088
	0·6				27·828				29·060
1	0	10·062	10·15	10·015	8·749	10·064	10·716	9·860	5·507
	0·2		10·499	10·497	9·217		11·170	10·564	6·332
	0·4			14·613	12·787			16·602	9·461
	0·6				28·045				29·248
1.25	0	10·487	10·374	10·340	9·028	10·542	11·084	10·115	5·671
	0·2		10·951	10·905	9·655		11·589	10·997	6·709
	0·4			14·926	13·132			16·875	9·809
	0·6				28·258				29·433
1·50	0	10·866	10·710	10·606	9·279	10·963	11·434	10·348	5·820
	0·2		11·364	11·268	10·036		11·984	11·376	7·025
	0·4			15·229	13·458			17·141	10·135
	0·6				28·470				26·617

Table 1 depicts fundamental frequency coefficients $\Omega_1 = \sqrt{(\rho h_0/D_{r0})}\omega_1 a^2$ for a circular annular plate of cylindrical anisotropy when the outer boundary is simply supported while Table 2 deals with the clamped case. The eigenvalues are tabulated as a function of r_b, r_c and e.

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